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SECOND GRADIENT OF STRAIN AND SURFACE-TENSION IN LINEAR ELASTICITY

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Abstract—In this paper there is formulated a linear theory of deformation of an elastic solid in which the potential energy-density is a function of the strain and its first and second gradients. This is a theory in which cohesive force and surface-tension are intrinsic. A solution is given for the strain and surface-tension, or surface-energy per unit area, resulting from separation of a solid along a plane; and a comparison is made with an analogous lattice model. Also presented are a general solution of the displacement-equation of equilibrium in terms of stress functions and the particular solution for the concentrated force. The special case of a liquid is considered and the solutions are given for the surface-tensions at plane and spherical surfaces.

1. INTRODUCTION

IF THE potential energy–density of an elastic continuum is assumed to depend on the rotation-gradient, in addition to the strain, there results a theory of elasticity with couple-stresses, i.e. couples per unit area, in addition to the usual stresses. In a paper on elastic materials with couple-stresses [1], R. A. Toupin observed that the rotation-gradient comprises only eight of the eighteen components of the strain-gradient and he proceeded to establish the mathematical theory in which all the components are accounted for. The additional components of the strain-gradient are accompanied by ten components of self-equilibrating double forces per unit area, i.e. stresses contributing neither resultant force nor couple, per unit area, across a surface in the material but, nevertheless, contributing to the potential energy and to the boundary conditions. In a subsequent paper, Toupin and Gazis [2] exhibited the correspondence between strain-gradient theory and atomic lattices with nearest neighbor and next nearest neighbor interactions; and they explored the consequences of an initial, homogeneous, self-equilibrating stress. They showed that a free surface will draw in or push out in a surface-layer; but only in non-centrosymmetric materials. Toupin pointed out, in a private conversation, that this restriction could be removed by taking into account, further, the second gradient of the strain. A start in this direction was made in 1959 by Hart [3] for liquids. Also, Green and Rivlin [4] have recently established the basis of a very general theory which includes strain-gradients of any order. It seems worthwhile, however, to formulate completely the equations of the simplest theory, incorporating the second gradient of strain, and to exhibit some typical solutions. The cohesive force and surface-tension, or surface-energy per unit area, which appear, may have some bearing on criteria of failure of solids and on a variety of surface-phenomena observed in both solids and liquids.

In what follows, equations of equilibrium, boundary conditions and constitutive equations are derived for a linear theory of elastic materials in which the potential energy-density is a 2nd degree polynomial in the infinitesimal strain and its first and second gradients. The case of the homogeneous, isotropic, centrosymmetric material is

explored in some detail. Terms appear which describe the property of cohesion and a general formula is given for surface-tension: one-half the product of a modulus of cohesion and the normal gradient of the dilatation at the surface. After a formulation of the general displacement-equation of equilibrium, a solution is given for the problem of separation of a solid along a plane, including formulas for the surface-energy per unit area and for the residual strain—which decays exponentially into the interior. A decay constant of the order of magnitude of interatomic distances is estimated from electron diffraction data obtained by Germer, MacRae and Hartman [5]. The theory admits either monotonic or oscillatory decay: a property which is shown to be also the case in a solution of the analogous problem for a monatomic lattice with first, second and third neighbor interactions. Following this, a general solution of the displacement-equation of equilibrium is given in terms of stress functions. Then the particular solution is found for the concentrated force. Finally, a special theory for a liquid is formulated by restricting the strain and its first two gradients to the dilatation and its first two gradients. Solutions are given for the surface-tensions at plane, and spherical, liquid surfaces. It is found that, as the curvature of the surface increases (decreases) from zero, the surface-tension becomes algebraically smaller (larger).

2. STRESS-EQUATION OF EQUILIBRIUM AND BOUNDARY CONDITIONS

In the linear theory to be formulated, the potential energy-density, W, is assumed to be a function of three polyadics:

$$W = W(\overset{1}{\boldsymbol{\varepsilon}}, \overset{2}{\boldsymbol{\varepsilon}}, \overset{3}{\boldsymbol{\varepsilon}}), \tag{1}$$

where

$$\overset{1}{\boldsymbol{\varepsilon}} = \tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla), \qquad \overset{2}{\boldsymbol{\varepsilon}} = \nabla\nabla\mathbf{u}, \qquad \overset{3}{\boldsymbol{\varepsilon}} = \nabla\nabla\nabla\mathbf{u}. \tag{2}$$

In (2), ∇ is the gradient operator and $\mathbf{u}$ is the displacement. The symmetric dyadic $\overset{1}{\boldsymbol{\varepsilon}}$ is the classical infinitesimal strain with six independent components. The triadic $\overset{2}{\boldsymbol{\varepsilon}}$, symmetric in the first two positions, has eighteen independent components which could be replaced [6] by the strain-gradient $\nabla\overset{1}{\boldsymbol{\varepsilon}}$, with eighteen independent components which are linear combinations of those of $\overset{2}{\boldsymbol{\varepsilon}}$, or by the rotation-gradient and a symmetric triadic with, respectively, eight and ten independent components which are linear combinations of those of $\overset{2}{\boldsymbol{\varepsilon}}$ or of $\nabla\overset{1}{\boldsymbol{\varepsilon}}$. Similarly, $\overset{3}{\boldsymbol{\varepsilon}}$, symmetric in the first three positions, has thirty independent components which could be replaced by the second gradient of strain $\nabla\nabla\overset{1}{\boldsymbol{\varepsilon}}$ or by a variety of combinations of other polyadics. Thus, the energy-density could be expressed as any one of a large number of functions of fifty-four independent variables. All of these energy functions lead to the same displacement-equation of equilibrium. The function chosen in (1) produces simpler forms of the boundary conditions and constitutive equations than most and as simple forms as any.

The variation of the total potential energy, in a volume V, with arbitrary variation of $\mathbf{u}$, is

$$\delta\int_V W\,\mathrm{d}V = \int_V (\overset{1}{\boldsymbol{\tau}} : \nabla\delta\mathbf{u} + \overset{2}{\boldsymbol{\tau}} \vdots \nabla\nabla\delta\mathbf{u} + \overset{3}{\boldsymbol{\tau}} :: \nabla\nabla\nabla\delta\mathbf{u})\,\mathrm{d}V, \tag{3}$$

where

$$\overset{1}{\boldsymbol{\tau}} = \frac{\partial W}{\partial \overset{1}{\boldsymbol{\varepsilon}}}, \qquad \overset{2}{\boldsymbol{\tau}} = \frac{\partial W}{\partial \overset{2}{\boldsymbol{\varepsilon}}}, \qquad \overset{3}{\boldsymbol{\tau}} = \frac{\partial W}{\partial \overset{3}{\boldsymbol{\varepsilon}}} \tag{4}$$

and the rule for multiple scalar multiplication of dyads, triads, etc. is

$$\mathbf{ab} : \mathbf{cd} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}), \qquad \mathbf{abc} \vdots \mathbf{def} = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{e})(\mathbf{c}\cdot\mathbf{f}), \qquad \text{etc.}$$

In the definitions (4), it is to be understood that $\overset{1}{\boldsymbol{\tau}}$, $\overset{2}{\boldsymbol{\tau}}$, $\overset{3}{\boldsymbol{\tau}}$ have the same symmetries as $\overset{1}{\boldsymbol{\varepsilon}}$, $\overset{2}{\boldsymbol{\varepsilon}}$, $\overset{3}{\boldsymbol{\varepsilon}}$, respectively.

By application of the chain rule of differentiation and the divergence theorem, the right-hand side of (3) is converted into the sum of

$$-\int_V [\nabla\cdot(\overset{1}{\boldsymbol{\tau}}-\nabla\cdot\overset{2}{\boldsymbol{\tau}}+\nabla\nabla:\overset{3}{\boldsymbol{\tau}})]\cdot\delta\mathbf{u}\,dV+\int_S \mathbf{n}\cdot(\overset{1}{\boldsymbol{\tau}}-\nabla\cdot\overset{2}{\boldsymbol{\tau}}+\nabla\nabla:\overset{3}{\boldsymbol{\tau}})\cdot\delta\mathbf{u}\,dS \tag{5}$$

and

$$\int_S \mathbf{n}\cdot(\overset{2}{\boldsymbol{\tau}}-\nabla\cdot\overset{3}{\boldsymbol{\tau}}):\nabla\delta\mathbf{u}\,dS+\int_S \mathbf{n}\cdot\overset{3}{\boldsymbol{\tau}}\vdots\nabla\nabla\delta\mathbf{u}\,dS, \tag{6}$$

where S is the boundary of V and $\mathbf{n}$ is the unit outward normal to S. Now, $\nabla\delta\mathbf{u}$ is not independent of $\delta\mathbf{u}$ on S because, if $\delta\mathbf{u}$ is known on S, so is the surface-gradient of $\delta\mathbf{u}$. This is analogous to the situation in couple-stress theory [7] where, if the displacement is known on S, so is the normal component of rotation—just as, in the classical theory of flexure of plates, if the deflection at the edge is known, so is the component of rotation about an axis normal to the edge. To prepare for the formulation of a variational principle, we resolve $\nabla\delta\mathbf{u}$ into a surface-gradient and a normal gradient:

$$\nabla\delta\mathbf{u} = \overset{s}{\nabla}\delta\mathbf{u}+\mathbf{n}D\delta\mathbf{u}; \qquad \overset{s}{\nabla} = (\mathbf{I}-\mathbf{nn})\cdot\nabla, \qquad D = \mathbf{n}\cdot\nabla, \tag{7}$$

where $\mathbf{I}$ is the unit dyadic. The term, in the integrand, with $\overset{s}{\nabla}\delta\mathbf{u}$ as coefficient can then be reduced to one with $\delta\mathbf{u}$ as coefficient by employing the chain rule and a surface integration—leaving only the independent variations $\delta\mathbf{u}$ and $D\delta\mathbf{u}$ as coefficients. A similar procedure, with some additional steps, can be used to convert the term with $\nabla\nabla\delta\mathbf{u}$ as coefficient to terms with the independent variations $\delta\mathbf{u}$, $D\delta\mathbf{u}$ and $D^2\delta\mathbf{u}$ as coefficients. Thus, if $\boldsymbol{\Phi}$ is a dyadic and $\mathbf{v}$ is a vector,

$$\boldsymbol{\Phi}:\nabla\mathbf{v} = \boldsymbol{\Phi}:\overset{s}{\nabla}\mathbf{v}+\mathbf{n}\cdot\boldsymbol{\Phi}\cdot D\mathbf{v} = \overset{s}{\nabla}\cdot(\boldsymbol{\Phi}\cdot\mathbf{v})-(\overset{s}{\nabla}\cdot\boldsymbol{\Phi})\cdot\mathbf{v}+\mathbf{n}\cdot\boldsymbol{\Phi}\cdot D\mathbf{v}.$$

By the surface divergence theorem [8] for a smooth, closed surface (which S is assumed to be),

$$\int_S \overset{s}{\nabla}\cdot(\boldsymbol{\Phi}\cdot\mathbf{v})\,dS = \int_S (\overset{s}{\nabla}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\Phi}\cdot\mathbf{v}\,dS.$$

Hence,

$$\int_S \boldsymbol{\Phi}:\nabla\mathbf{v}\,dS = \int_S (\mathbf{L}\cdot\boldsymbol{\Phi})\cdot\mathbf{v}\,dS+\int_S \mathbf{n}\cdot\boldsymbol{\Phi}\cdot D\mathbf{v}\,dS, \tag{8}$$

where

$$\mathbf{L} = \mathbf{n}\overset{s}{\nabla}\cdot\mathbf{n}-\overset{s}{\nabla}. \tag{9}$$

Similarly, if $\boldsymbol{\Psi}$ is a triadic,

$$\int_S \boldsymbol{\Psi}\vdots\nabla\boldsymbol{\Phi}\,dS = \int_S (\mathbf{L}\cdot\boldsymbol{\Psi}):\boldsymbol{\Phi}\,dS+\int_S \mathbf{n}\cdot\boldsymbol{\Psi}:D\boldsymbol{\Phi}\,dS. \tag{10}$$

Thus, by (8), the first integral in (6) can be written as

$$\int_S \mathbf{L}\cdot[\mathbf{n}\cdot(\overset{2}{\tau}-\nabla\cdot\overset{3}{\tau})]\cdot\delta\mathbf{u}\,dS+\int_S \mathbf{nn}:(\overset{2}{\tau}-\nabla\cdot\overset{3}{\tau})\cdot D\delta\mathbf{u}\,dS \tag{11}$$

and, by (10), the second integral in (6) is

$$\int_S \mathbf{L}\cdot(\mathbf{n}\cdot\overset{3}{\tau}):\nabla\delta\mathbf{u}\,dS+\int_S \mathbf{nn}:\overset{3}{\tau}:D\nabla\delta\mathbf{u}\,dS. \tag{12}$$

Again by (8), the first integral in (12) can be written as

$$\int_S \mathbf{L}\cdot[\mathbf{L}\cdot(\mathbf{n}\cdot\overset{3}{\tau})]\cdot\delta\mathbf{u}\,dS+\int_S \mathbf{n}\cdot[\mathbf{L}\cdot(\mathbf{n}\cdot\overset{3}{\tau})]\cdot D\delta\mathbf{u}\,dS. \tag{13}$$

As for the second integral in (12), we suppose that S is a coordinate surface of an orthogonal, curvilinear, coordinate system and interpret $\mathbf{n}$ as one of the orthogonal unit vectors in that system. Then

$$D\nabla\mathbf{v}=\mathbf{n}\cdot\nabla\nabla\mathbf{v}=\nabla(\mathbf{n}\cdot\nabla\mathbf{v})-(\nabla\mathbf{n})\cdot\nabla\mathbf{v}.$$

But $\nabla(\mathbf{n}\cdot\nabla\mathbf{v})=\nabla D\mathbf{v}=\overset{s}{\nabla}D\mathbf{v}+\mathbf{n}D^2\mathbf{v}$ and $(\nabla\mathbf{n})\cdot\nabla\mathbf{v}=(\overset{s}{\nabla}\mathbf{n})\cdot\overset{s}{\nabla}\mathbf{v}$. Hence:

$$D\nabla\mathbf{v}=\overset{s}{\nabla}D\mathbf{v}+\mathbf{n}D^2\mathbf{v}-(\overset{s}{\nabla}\mathbf{n})\cdot\overset{s}{\nabla}\mathbf{v}.$$

Accordingly,

$$\begin{aligned}\boldsymbol{\Phi}:D\nabla\mathbf{v}&=\boldsymbol{\Phi}:[\overset{s}{\nabla}D\mathbf{v}-(\overset{s}{\nabla}\mathbf{n})\cdot\overset{s}{\nabla}\mathbf{v}+\mathbf{n}D^2\mathbf{v}],\\&=\overset{s}{\nabla}\cdot[\boldsymbol{\Phi}\cdot D\mathbf{v}-(\overset{s}{\nabla}\mathbf{n})\cdot\boldsymbol{\Phi}\cdot\mathbf{v}]-(\overset{s}{\nabla}\cdot\boldsymbol{\Phi})\cdot D\mathbf{v}+\overset{s}{\nabla}\cdot[(\overset{s}{\nabla}\mathbf{n})\cdot\boldsymbol{\Phi}]\cdot\mathbf{v}+\mathbf{n}\cdot\boldsymbol{\Phi}\cdot D^2\mathbf{v},\\&=\overset{s}{\nabla}\cdot[\boldsymbol{\Phi}\cdot D\mathbf{v}-(\overset{s}{\nabla}\mathbf{n})\cdot\boldsymbol{\Phi}\cdot\mathbf{v}]-(\overset{s}{\nabla}\cdot\boldsymbol{\Phi})\cdot D\mathbf{v}-\mathbf{L}\cdot[(\overset{s}{\nabla}\mathbf{n})\cdot\boldsymbol{\Phi}]\cdot\mathbf{v}+\mathbf{n}\cdot\boldsymbol{\Phi}\cdot D^2\mathbf{v},\end{aligned}$$

where the last step is a consequence of $\mathbf{n}\cdot(\overset{s}{\nabla}\mathbf{n})=0$ and (9). Hence, using the surface divergence theorem and noting again that $\mathbf{n}\cdot(\overset{s}{\nabla}\mathbf{n})=0$, we can write the second integral in (12) as

$$-\int_S \mathbf{L}\cdot[(\overset{s}{\nabla}\mathbf{n})\cdot(\mathbf{nn}:\overset{3}{\tau})]\cdot\delta\mathbf{u}\,dS+\int_S \mathbf{L}\cdot(\mathbf{nn}:\overset{3}{\tau})\,D\delta\mathbf{u}\,dS+\int_S \mathbf{nnn}:\overset{3}{\tau}\cdot D^2\delta\mathbf{u}\,dS. \tag{14}$$

Altogether, the right-hand side of (3) is the sum of (5), (11), (13) and (14); so that (3) becomes

$$\begin{aligned}\delta\int_V W\,dV=&-\int_V[\nabla\cdot(\overset{1}{\tau}-\nabla\cdot\overset{2}{\tau}+\nabla\nabla:\overset{3}{\tau})]\cdot\delta\mathbf{u}\,dV+\int_S\mathbf{n}\cdot(\overset{1}{\tau}-\nabla\cdot\overset{2}{\tau}+\nabla\nabla:\overset{3}{\tau})\cdot\delta\mathbf{u}\,dS\\&+\int_S\{\mathbf{L}\cdot[\mathbf{n}\cdot(\overset{2}{\tau}-\nabla\cdot\overset{3}{\tau})+\mathbf{L}\cdot(\mathbf{n}\cdot\overset{3}{\tau})-(\overset{s}{\nabla}\mathbf{n})\cdot(\mathbf{nn}:\overset{3}{\tau})]\}\cdot\delta\mathbf{u}\,dS\\&+\int_S\{\mathbf{nn}:(\overset{2}{\tau}-\nabla\cdot\overset{3}{\tau})+\mathbf{n}\cdot[\mathbf{L}\cdot(\mathbf{n}\cdot\overset{3}{\tau})]+\mathbf{L}\cdot(\mathbf{nn}:\overset{3}{\tau})\}\cdot D\delta\mathbf{u}\,dS\\&+\int_S(\mathbf{nnn}:\overset{3}{\tau})\cdot D^2\delta\mathbf{u}\,dS.\end{aligned} \tag{15}$$

We now assume the following principle of stationary potential energy:

$$\delta \int_V W \, dV = \int_V \mathbf{f} \cdot \delta \mathbf{u} \, dV + \int_S (\overset{1}{\mathbf{t}} \cdot \delta \mathbf{u} + \overset{2}{\mathbf{t}} \cdot D\delta \mathbf{u} + \overset{3}{\mathbf{t}} \cdot D^2 \delta \mathbf{u}) \, dS, \tag{16}$$

where $\mathbf{f}$ is the body force per unit volume and $\overset{1}{\mathbf{t}}$, $\overset{2}{\mathbf{t}}$, $\overset{3}{\mathbf{t}}$ are generalized surface tractions. From (15) and (16) there ensues the stress-equation of equilibrium

$$\nabla \cdot (\overset{1}{\boldsymbol{\tau}} - \nabla \cdot \overset{2}{\boldsymbol{\tau}} + \nabla\nabla : \overset{3}{\boldsymbol{\tau}}) + \mathbf{f} = 0 \tag{17}$$

and the traction boundary conditions

$$\overset{1}{\mathbf{t}} = \mathbf{n} \cdot (\overset{1}{\boldsymbol{\tau}} - \nabla \cdot \overset{2}{\boldsymbol{\tau}} + \nabla\nabla : \overset{3}{\boldsymbol{\tau}}) + \mathbf{L} \cdot [\mathbf{n} \cdot (\overset{2}{\boldsymbol{\tau}} - \nabla \cdot \overset{3}{\boldsymbol{\tau}}) + \mathbf{L} \cdot (\mathbf{n} \cdot \overset{3}{\boldsymbol{\tau}}) - (\overset{s}{\nabla}\mathbf{n}) \cdot (\mathbf{nn} : \overset{3}{\boldsymbol{\tau}})], \tag{18a}$$

$$\overset{2}{\mathbf{t}} = \mathbf{nn} : (\overset{2}{\boldsymbol{\tau}} - \nabla \cdot \overset{3}{\boldsymbol{\tau}}) + \mathbf{n} \cdot [\mathbf{L} \cdot (\mathbf{n} \cdot \overset{3}{\boldsymbol{\tau}})] + \mathbf{L} \cdot (\mathbf{nn} : \overset{3}{\boldsymbol{\tau}}), \tag{18b}$$

$$\overset{3}{\mathbf{t}} = \mathbf{nnn} \vdots \overset{3}{\boldsymbol{\tau}}. \tag{18c}$$

Thus, there are three scalar equations of equilibrium and only nine scalar boundary conditions instead of what appeared at first, in (5) and (6), to be $3+9+18 = 30$ boundary conditions. This reduction is analogous to the reduction from three to two boundary conditions in the classical theory of flexure of plates.

Other admissible sets of nine boundary conditions, in terms of $\mathbf{u}$, $D\mathbf{u}$ and $D^2\mathbf{u}$ or products of appropriate components of $\overset{1}{\mathbf{t}}$ and $\mathbf{u}$ or $\overset{2}{\mathbf{t}}$ and $D\mathbf{u}$ or $\overset{3}{\mathbf{t}}$ and $D^2\mathbf{u}$, are apparent from the form of (16) and the uniqueness theorem which would follow the steps from (1) to (17) in reverse.

3. CONSTITUTIVE EQUATIONS

We shall consider, here, only homogeneous, centrosymmetric, isotropic materials. Then, in the 2nd degree polynomial W, with constant coefficients, the variables $\overset{1}{\boldsymbol{\varepsilon}}$, $\overset{2}{\boldsymbol{\varepsilon}}$, $\overset{3}{\boldsymbol{\varepsilon}}$ can appear in the linear terms only as scalars and in the quadratic terms only as products that are scalars. This eliminates the terms linear in $\overset{2}{\boldsymbol{\varepsilon}}$ and the products of $\overset{2}{\boldsymbol{\varepsilon}}$ with $\overset{1}{\boldsymbol{\varepsilon}}$ and $\overset{3}{\boldsymbol{\varepsilon}}$. Also, there are only two quadratic products of $\overset{1}{\boldsymbol{\varepsilon}}$, five of $\overset{2}{\boldsymbol{\varepsilon}}$, seven of $\overset{3}{\boldsymbol{\varepsilon}}$ and three of $\overset{1}{\boldsymbol{\varepsilon}}$ with $\overset{3}{\boldsymbol{\varepsilon}}$ that are scalars. Furthermore, we may omit the constant term and the term linear in $\overset{1}{\boldsymbol{\varepsilon}}$ because, just as in classical elasticity, the material configuration, to which W is referred, can be chosen so that these terms do not appear. What remains, then, is

$$\begin{aligned} W = {} & \tfrac{1}{2}\lambda \varepsilon_{ii}\varepsilon_{jj} + \mu \varepsilon_{ij}\varepsilon_{ij} + a_1 \varepsilon_{ijj}\varepsilon_{ikk} + a_2 \varepsilon_{iik}\varepsilon_{kjj} + a_3 \varepsilon_{iik}\varepsilon_{jjk} \\ & + a_4 \varepsilon_{ijk}\varepsilon_{ijk} + a_5 \varepsilon_{ijk}\varepsilon_{kji} + b_1 \varepsilon_{iij}\varepsilon_{kkll} + b_2 \varepsilon_{ijkk}\varepsilon_{ijll} \\ & + b_3 \varepsilon_{iijk}\varepsilon_{jkll} + b_4 \varepsilon_{iijk}\varepsilon_{llkj} + b_5 \varepsilon_{iijk}\varepsilon_{lljk} + b_6 \varepsilon_{ijkl}\varepsilon_{ijkl} \\ & + b_7 \varepsilon_{ijkl}\varepsilon_{jkli} + c_1 \varepsilon_{ii}\varepsilon_{jjkk} + c_2 \varepsilon_{ij}\varepsilon_{ijkk} + c_3 \varepsilon_{ij}\varepsilon_{kkij} + b_0 \varepsilon_{iijj}, \end{aligned} \tag{19}$$

where ε_{ij}, ε_{ijk}, ε_{ijkl} $(i, j, k, l = 1, 2, 3)$ are the components of $\overset{1}{\boldsymbol{\varepsilon}}$, $\overset{2}{\boldsymbol{\varepsilon}}$, $\overset{3}{\boldsymbol{\varepsilon}}$, respectively, in any orthogonal coordinate system, and the summation convention for repeated indices is employed.

The constants λ and μ are the usual Lamé constants and the five a_n are the additional constants which appear in Toupin's strain-gradient theory [1, 6].

The choice of the polynomial (19) implies that terms of higher degree are small in comparison with those retained. This, in turn, implies that certain assumptions as to order of magnitude have been made in addition to $|\nabla \mathbf{u}| \ll 1$ of the classical linear theory. While $\overset{1}{\boldsymbol{\varepsilon}}$ is dimensionless, $\overset{2}{\boldsymbol{\varepsilon}}$ has the dimension of reciprocal of length and $\overset{3}{\boldsymbol{\varepsilon}}$ has the dimension of reciprocal of the square of length. Thus, the ratios b_n/μ have the dimension of the fourth power of length and the ratios a_n/μ, b_0/μ, c_n/μ have the dimension of square of length. Suppose that l^2 is the ratio of one of the constants a_n to μ; i.e. l is a material property with the dimension of length. Then we assume that $a_n/\mu l^2$, $b_n/\mu l^4$, $c_n/\mu l^2$, $b_0/\mu l^2$ are of order of magnitude unity and

$$|\nabla \mathbf{u}| \ll 1, \qquad l|\nabla\nabla \mathbf{u}| \ll 1, \qquad l^2|\nabla\nabla\nabla \mathbf{u}| \ll 1,$$

where $|\boldsymbol{\Psi}|$ signifies the absolute values of the components of $\boldsymbol{\Psi}$. With these assumptions, all of the terms in (19) are of the same order of magnitude and terms of higher degree are negligibly small.

From (19) and (4) follow the constitutive equations:

$$\tau_{pq} = \lambda\varepsilon_{ii}\delta_{pq} + 2\mu\varepsilon_{pq} + c_1\varepsilon_{iijj}\delta_{pq} + c_2\varepsilon_{pqii} + \tfrac{1}{2}c_3(\varepsilon_{iipq} + \varepsilon_{iiqp}), \tag{20a}$$

$$\begin{aligned}\tau_{pqr} = {} & a_1(\varepsilon_{pii}\delta_{qr} + \varepsilon_{qii}\delta_{pr}) + \tfrac{1}{2}a_2(\varepsilon_{iip}\delta_{qr} + 2\varepsilon_{rii}\delta_{qp} + \varepsilon_{iiq}\delta_{pr}) \\ & + 2a_3\varepsilon_{iir}\delta_{pq} + 2a_4\varepsilon_{pqr} + a_5(\varepsilon_{rqp} + \varepsilon_{rpq}),\end{aligned} \tag{20b}$$

$$\begin{aligned}\tau_{pqrs} = {} & \tfrac{2}{3}b_1\varepsilon_{iijj}\delta_{pqrs} + \tfrac{2}{3}b_2\varepsilon_{jkii}\delta_{jkpqrs} + \tfrac{1}{6}b_3[(\varepsilon_{iijk} + \varepsilon_{iikj})\delta_{jkpqrs} + 2\varepsilon_{jsii}\delta_{jpqr}] \\ & + \tfrac{2}{3}b_4\varepsilon_{iisj}\delta_{jpqr} + \tfrac{2}{3}b_5\varepsilon_{iijs}\delta_{jpqr} + 2b_6\varepsilon_{pqrs} + \tfrac{2}{3}b_7(\varepsilon_{qrsp} + \varepsilon_{rspq} + \varepsilon_{spqr}) \\ & + \tfrac{1}{3}c_1\varepsilon_{ii}\delta_{pqrs} + \tfrac{1}{3}c_2\varepsilon_{ij}\delta_{ijpqrs} + \tfrac{1}{3}c_3\varepsilon_{is}\delta_{ipqr} + \tfrac{1}{3}b_0\delta_{pqrs},\end{aligned} \tag{20c}$$

where τ_{pq}, τ_{pqr}, τ_{pqrs} are the components of $\overset{1}{\boldsymbol{\tau}}$, $\overset{2}{\boldsymbol{\tau}}$, $\overset{3}{\boldsymbol{\tau}}$, respectively; δ_{ij} is the Kronecker delta and

$$\delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}, \qquad \delta_{ijklmn} = \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jm}\delta_{kn}.$$

The τ_{pq} are like the components of the usual stress with the dimensions of force per unit area; but here they depend on the second gradient of strain in addition to the strain. The τ_{pqr} have the character of double forces per unit area. The combinations $\frac{1}{2}(\tau_{pqr} - \tau_{prq})$ are the eight components of the deviator of the couple-stress, i.e. couples per unit area, while the remaining ten independent combinations are self-equilibrating. The thirty τ_{pqrs} have the dimensions of force and the character of triple forces per unit area—all self-equilibrating.

4. SURFACE-TENSION

Of special interest, in the energy–density (19), is the term linear in $\overset{3}{\boldsymbol{\varepsilon}}$ which produces the homogeneous, self-equilibrating components of *cohesive force*, $\frac{1}{3}b_0\delta_{pqrs}$, in (20c). The coefficient b_0, with the dimensions of force, is a *modulus of cohesion*. It gives rise to surface-tension; or, equivalently, to the energy, per unit area, associated with the formation of a new surface.

In classical hydrostatics, surface-tension is introduced by postulating the existence of a vanishingly thin surface-membrane under uniform tension. If the liquid is under no external forces, the surface-tension is equal to the total potential energy divided by the

surface-area. In a solid, the surface-tension in a body under no external forces need not be uniform; but we may adopt the same definition for the average surface-tension:

$$T_{\text{ave}} = \mathscr{W}/A, \tag{21}$$

where A is the area of the surface and

$$\mathscr{W} = \int_V W \, \mathrm{d}V;$$

or, from (19) and (20),

$$\mathscr{W} = \tfrac{1}{2}\int_V (\overset{1}{\boldsymbol{\tau}} : \nabla\mathbf{u} + \overset{2}{\boldsymbol{\tau}} \vdots \nabla\nabla\mathbf{u} + \overset{3}{\boldsymbol{\tau}} :: \nabla\nabla\nabla\mathbf{u}) \, \mathrm{d}V + \tfrac{1}{2}b_0 \int_V \nabla^2\nabla \cdot \mathbf{u} \, \mathrm{d}V. \tag{22}$$

The second integral in (22) enters because the cohesive force is constant whereas the remaining parts of the "stresses" vary in proportion to the strain and its gradients.

Now, if there are no external forces, i.e. if $\overset{1}{\mathbf{t}}, \overset{2}{\mathbf{t}}, \overset{3}{\mathbf{t}}$ and $\mathbf{f}$ are zero, the first integral in (22) vanishes; as can be shown by the procedure employed in Section 2. Hence,

$$T_{\text{ave}} = \frac{b_0}{2A}\int_V \nabla^2\nabla \cdot \mathbf{u} \, \mathrm{d}V = \frac{b_0}{2A}\int_S \mathbf{n} \cdot \nabla\nabla \cdot \mathbf{u} \, \mathrm{d}S.$$

With this result, it seems appropriate to take

$$T = \tfrac{1}{2}b_0 \mathbf{n} \cdot \nabla\nabla \cdot \mathbf{u}|_S \tag{23}$$

as the definition of the *punctual* surface-tension, or surface-energy per unit area, in a centrosymmetric, isotropic, elastic solid; i.e. one-half the product of the modulus of cohesion and the normal gradient of the dilatation at the surface.

5. DISPLACEMENT-EQUATION OF EQUILIBRIUM

Upon substituting the strain–displacement relations (2) into the stress–strain relations (20) and the resulting expressions for the stresses into the stress-equation of equilibrium (17), we find the displacement-equation of equilibrium

$$[\lambda + 2\mu - (\bar{a} - 2\bar{c})\nabla^2 + \bar{b}\nabla^4]\nabla\nabla \cdot \mathbf{u} - [\mu - (\bar{a}' - c_3)\nabla^2 + \bar{b}'\nabla^4]\nabla \times \nabla \times \mathbf{u} + \mathbf{f} = 0, \tag{24}$$

where

$$\bar{a} = 2(a_1 + a_2 + a_3 + a_4 + a_5), \quad \bar{b} = 2(b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7),$$
$$\bar{c} = c_1 + c_2 + c_3, \quad \bar{a}' = 2(a_3 + a_4), \quad \bar{b}' = 2(b_5 + b_6).$$

Alternatively, (24) may be written in the form

$$(\lambda + 2\mu)D_{11}^2 D_{12}^2 \nabla\nabla \cdot \mathbf{u} - \mu D_{21}^2 D_{22}^2 \nabla \times \nabla \times \mathbf{u} + \mathbf{f} = 0, \tag{25}$$

where

$$D_{ij}^2 = 1 - l_{ij}^2\nabla^2, \qquad i = 1, 2; \qquad j = 1, 2; \tag{26}$$

and

$$2(\lambda + 2\mu)l_{1j}^2 = \bar{a} - 2\bar{c} \pm [(\bar{a} - 2\bar{c})^2 - 4\bar{b}(\lambda + 2\mu)]^{\frac{1}{2}}, \qquad j = 1, 2; \tag{27a}$$

$$2\mu l_{2j}^2 = \bar{a}' - c_3 \pm [(\bar{a}' - c_3)^2 - 4\bar{b}'\mu]^{\frac{1}{2}}, \qquad j = 1, 2. \tag{27b}$$

Thus, in general, solutions of the equation of equilibrium will contain Lamé's constants, λ and μ, and four additional material constants, l_{ij}, having the dimension of length. Application of boundary conditions may introduce some or all of the remaining material constants which appear in the energy-density (19).

The conditions for positive W do not include relations between $\bar{a}$ (or $\bar{a}'$) and $\bar{b}, \bar{c}$ (or $\bar{b}', c_3$) and, hence, supply no indication of the character, real or complex, of the l_{ij}. In what follows, the l_{ij} will be treated as if they were real and positive; but complex l_{ij} are equally admissible.

6. STRAIN AND SURFACE-TENSION AT A PLANE SURFACE OF A SOLID

In a rectangular coordinate system x, y, z, consider the half-space $x \geqslant 0$ without body force and with its plane surface traction-free:

$$\overset{1}{\mathbf{t}} = \overset{2}{\mathbf{t}} = \overset{3}{\mathbf{t}} = 0 \qquad \text{on } x = 0. \tag{28}$$

Assume

$$u_x = u_x(x), \qquad u_y = u_z = 0. \tag{29}$$

Then, with primes designating differentiations with respect to x, the stress-equation of equilibrium (17) reduces to

$$\tau'_{xx} - \tau''_{xxx} + \tau'''_{xxxx} = 0, \tag{30}$$

while the boundary conditions (18) become

$$(\tau_{xx} - \tau'_{xxx} + \tau''_{xxxx})_{x=0} = (\tau_{xxx} - \tau'_{xxxx})_{x=0} = (\tau_{xxxx})_{x=0} = 0. \tag{31}$$

Also, with the assumed form of displacement (29), the constitutive equations (20) yield

$$\tau_{xx} = (\lambda + 2\mu)\varepsilon + \bar{c}\varepsilon'', \qquad \tau_{xxx} = \bar{a}\varepsilon', \qquad \tau_{xxxx} = b_0 + \bar{c}\varepsilon + \bar{b}\varepsilon'', \tag{32}$$

where $\varepsilon = \mathrm{d}u_x/\mathrm{d}x$. Further, the displacement-equation of equilibrium (25) becomes

$$\left(1 - l_{11}^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\left(1 - l_{12}^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\frac{\mathrm{d}^2 u_x}{\mathrm{d}x^2} = 0. \tag{33}$$

One integration of the stress-equation of equilibrium (30) makes $\tau_{xx} - \tau'_{xxx} + \tau''_{xxxx}$ equal to a constant which, by the first of the boundary conditions (31), must be zero. This disposes of one of the three boundary conditions on $x = 0$. The remaining two, expressed in terms of ε through (32), are

$$[(\bar{a} - \bar{c})\varepsilon' - \bar{b}\varepsilon''']_{x=0} = 0, \qquad (\bar{c}\varepsilon + \bar{b}\varepsilon'')_{x=0} = -b_0. \tag{34}$$

The solution of (33), vanishing at infinity, is

$$u_x = A_1 e^{-x/l_{11}} + A_2 e^{-x/l_{12}}. \tag{35}$$

Inserting (35) into (34) and making use of the definitions (27a), we find the following equations for determining the constants A_1 and A_2:

$$A_1 l_{12}^2 (l_{11}^2 + l_{10}^2) + A_2 l_{11}^2 (l_{12}^2 + l_{10}^2) = 0, \tag{36a}$$

$$A_1 l_{12} (l_{12}^2 + l_{10}^2) + A_2 l_{11} (l_{11}^2 + l_{10}^2) = b_0 l_{11} l_{12}/(\lambda + 2\mu), \tag{36b}$$

where $l_{10}^2 = \bar{c}/(\lambda + 2\mu)$. From (36), the values of A_1 and A_2 follow immediately.

The formula (23), for the surface-tension yields

$$T = -\tfrac{1}{2}b_0\varepsilon'|_{x=0} = -\tfrac{1}{2}b_0(A_1 l_{11}^{-2} + A_2 l_{12}^{-2}),$$

or

$$T = \frac{b_0^2(l_{11}^2 - l_{12}^2)}{2(\lambda+2\mu)[l_{11}(l_{12}^2+l_{10}^2)^2 - l_{12}(l_{11}^2+l_{10}^2)^2]}. \tag{37}$$

This is also, for each surface, the energy per unit area required to separate the body along a plane.

As may be seen from (35), the displacement and strain decay exponentially with distance from the surface. It is known that such an effect is confined to an extremely thin surface-layer in physical materials; so that l_{11} and l_{12} are probably very small lengths. An estimate of order of magnitude can be obtained from measurements of electron diffraction at nickel surfaces by Germer, MacRae and Hartman [5]. They conclude that the displacement of the superficial layer of atoms toward the interior is five times as large as that of the next layer. The assumption of a simple exponential decay, proportional to $e^{-x/l}$, would lead to an l of about five eighths of the distance between adjacent planes of atoms. This is a ratio of the same order of magnitude as the analogous one of about three eighths of the diameter of a sphere, in a simple cubic array of contiguous elastic spheres, computed for the case when only the rotation-gradient (leading to couple-stresses) is taken into account, in the energy-density, in addition to the strain [9]. If the l_{ij} are complex, the solution (35) would take the form of products of negative exponentials and trigonometric functions. Then the decay of displacement and strain would be oscillatory and the estimate of order of magnitude would apply to the real part.

7. LATTICE MODEL

Consider a one-dimensional lattice of interacting particles distributed along the x-axis at points $x = nh$, where n is a positive or negative integer and h is a constant which is taken equal to unity. Suppose that forces between particles are limited to first, second and third neighbors and include self-equilibrating initial forces. Now separate the lattice between particles $n = 0$ and $n = -1$ and consider the semi-infinite lattice $n \geqslant 0$. The separation is effected by the addition of forces P_0, P_1, P_2, on particles 0, 1, 2, respectively, equal and opposite to the resultants of the initial forces exerted on those particles by the particles $-1, -2, -3$. Since the initial forces are self-equilibrating,

$$P_0 + P_1 + P_2 = 0. \tag{38}$$

Now, suppose that the change of force between two particles is proportional to their relative displacement; with constants of proportionality $\alpha_1, \alpha_2, \alpha_3$ for first, second and third neighbors, respectively. Then, if u_n is the displacement of the nth particle, the equilibrium of that particle is expressed by

$$\sum_{i=1}^{i=3} \alpha_i(u_{n+i} - 2u_n + u_{n-i}) = 0, \qquad n \geqslant 3, \tag{39}$$

$$\sum_{i=1}^{i=2} \alpha_i(u_{2+i} - 2u_2 + u_{2-i}) + \alpha_3(u_5 - u_2) = P_2, \qquad n = 2, \tag{40a}$$

$$\alpha_1(u_2-2u_1+u_0)+\alpha_2(u_3-u_1)+\alpha_3(u_4-u_1) = P_1, \qquad n = 1, \tag{40b}$$

$$\alpha_1(u_1-u_0)+\alpha_2(u_2-u_0)+\alpha_3(u_3-u_0) = P_0, \qquad n = 0. \tag{40c}$$

If we adopt the notation $\Delta u_n = u_{n+1}-u_n$, we have

$$\Delta^2 u_n = u_{n+1}-2u_n+u_{n-1},$$

$$\Delta^4 u_n = u_{n+2}-4u_{n+1}+6u_n-4u_{n-1}+u_{n-2},$$

$$\Delta^6 u_n = u_{n+3}-6u_{n+2}+15u_{n+1}-20u_n+15u_{n-1}-6u_{n-2}+u_{n-3}.$$

Then, with

$$\beta_1 = \alpha_1+4\alpha_2+9\alpha_3, \qquad \beta_2 = -\alpha_2-6\alpha_3, \tag{41}$$

equation (39) can be written as

$$(\beta_1-\beta_2\Delta^2+\alpha_3\Delta^4)\Delta^2 u_n = 0; \tag{42}$$

or

$$(1-\lambda_1^2\Delta^2)(1-\lambda_2^2\Delta^2)\Delta^2 u_n = 0, \tag{43}$$

where

$$2\beta_1\lambda_i^2 = \beta_2 \pm (\beta_2^2-4\beta_1\alpha_3)^{\frac{1}{2}}, \qquad i = 1, 2. \tag{44}$$

The solution of (43), vanishing as $n \to \infty$, is

$$u_n = A_1' e^{-n/\mu_1}+A_2' e^{-n/\mu_2}, \tag{45}$$

where

$$\cosh \mu_i^{-1} = 1+\tfrac{1}{2}\lambda_i^{-2}, \qquad i = 1, 2. \tag{46}$$

As for the boundary conditions, again there are three for the two constants A_1' and A_2'. However, following Gazis and Wallis [10], we sum the three equations (40) and obtain

$$(1-\lambda_1^2\Delta^2)(1-\lambda_2^2\Delta^2)\Delta u_2 = 0, \tag{47}$$

which is satisfied identically by the solution (45). Thus, as in Section 6, the general condition of equilibrium disposes of one of the three boundary conditions. This leaves two conditions which may be any two, independent, linear combinations of the three conditions (40) other than their sum. The resulting formulas for A_1' and A_2' are lengthy expressions of little present interest.

More important is a comparison of the solutions (45), for the lattice, and (35) for the continuum. Both are comprised of two negative exponentials. In the case of the continuum, no conclusion could be reached regarding the character, real or complex, of the decay constants. The same is true for the lattice.

The quadratic part of the potential energy of the nth particle, $n > 2$, is

$$W_n = \tfrac{1}{2}\beta_1(\Delta u_n)^2+\tfrac{1}{2}\beta_2'(\Delta^2 u_n)^2+\tfrac{1}{2}\alpha_3(\Delta^3 u_n)^2+\beta_2''\Delta u_n\Delta^3 u_n;$$

i.e. an energy leading to the equilibrium equation (42):

$$-\frac{\partial W_n}{\partial u_n} = (\beta_1-\beta_2\Delta^2+\alpha_3\Delta^4)\Delta^2 u_n = 0,$$

where $\beta_2 = \beta'_2 - 2\beta''_2$. The conditions for positive W_n supply no relation between β'_2 (corresponding to $\bar{a}$ of the continuum) and the remaining constants; and, hence, no indication of the character of the λ_i and μ_i.

For real, positive force constants, α_i, standing in the relations

$$\alpha_1 \gg \alpha_2 \gg \alpha_3,$$

(for example, going inversely as high powers of the distances between interacting particles) the roots λ_i in (44) would be complex and then (46) would require complex μ_i. However, real, positive α_i, though appealing, are not necessary for stability.

8. STRESS FUNCTIONS

In this section, it is proved that any solution, $\mathbf{u}$, of the displacement-equation of equilibrium, (24) or (25), in a region V bounded by a surface S, can be expressed as

$$\mathbf{u} = \mathbf{B} - (l_{21}^2 + l_{22}^2 - l_{21}^2 l_{22}^2 \nabla^2)\nabla\nabla \cdot \mathbf{B} - \tfrac{1}{2}\nabla(D_{11}^2 D_{12}^2 - k^{-1})(\mathbf{r} \cdot D_{21}^2 D_{22}^2 \mathbf{B} + B_0), \tag{48}$$

where $k = (\lambda + 2\mu)/\mu$, $\mathbf{r}$ is the position vector and

$$\mu D_{11}^2 D_{12}^2 \nabla^2 B_0 = \mathbf{r} \cdot D_{11}^2 D_{12}^2 \mathbf{f} - 4(l_{11}^2 + l_{12}^2 - \tfrac{3}{2} l_{11}^2 l_{12}^2 \nabla^2)\nabla \cdot \mathbf{f}, \tag{49}$$

$$\mu D_{21}^2 D_{22}^2 \nabla^2 \mathbf{B} = -\mathbf{f}. \tag{50}$$

First, by the usual proof of the Helmholtz resolution, functions φ and $\mathbf{H}$ can be defined, in terms of $\mathbf{u}$, so that

$$\mathbf{u} = \nabla\varphi + \nabla \times \mathbf{H}, \qquad \nabla \cdot \mathbf{H} = 0. \tag{51}$$

Upon substituting (51) in (25), we find

$$\mu\nabla^2[kD_{11}^2 D_{12}^2 \nabla\varphi + D_{21}^2 D_{22}^2 \nabla \times \mathbf{H}] + \mathbf{f} = 0. \tag{52}$$

Define

$$4\pi\mathbf{B}' = \int_V \psi(r_1)[kD_{11}^2 D_{12}^2 \nabla\varphi + D_{21}^2 D_{22}^2 \nabla \times \mathbf{H}]_Q \, dV_Q, \tag{53}$$

where

$$\psi(r_1) = r_1^{-1}(e^{-r_1/l_{21}} - e^{-r_1/l_{22}})/(l_{21}^2 - l_{22}^2), \tag{54}$$

r_1 is the distance between a field point $P(x, y, z)$ and a source point $Q(\xi, \eta, \zeta)$ and $dV_Q = d\xi \, d\eta \, d\zeta$. Then, by a process similar to that for Poisson's equation [11],

$$D_{21}^2 D_{22}^2 \mathbf{B}' = kD_{11}^2 D_{12}^2 \nabla\varphi + D_{21}^2 D_{22}^2 \nabla \times \mathbf{H} \tag{55}$$

and, from (55) and (52),

$$\mu D_{21}^2 D_{22}^2 \nabla^2 \mathbf{B}' + \mathbf{f} = 0. \tag{56}$$

The divergence of (55) produces

$$D_{21}^2 D_{22}^2 \nabla \cdot \mathbf{B}' = kD_{11}^2 D_{12}^2 \nabla^2 \varphi. \tag{57}$$

Define

$$B_0 = 2k\varphi - \mathbf{r} \cdot D_{21}^2 D_{22}^2 \mathbf{B}' \tag{58}$$

and find

$$\mu D_{11}^2 D_{12}^2 \nabla^2 B_0 = \mathbf{r} \cdot D_{11}^2 D_{12}^2 \mathbf{f} - 4(l_{11}^2 + l_{12}^2 - \tfrac{3}{2} l_{11}^2 l_{12}^2 \nabla^2) \nabla \cdot \mathbf{f} \tag{59}$$

by using (56) and (57). Now, define

$$\mathbf{B}'' = \nabla \times \mathbf{H} - \mathbf{B}' + (l_{21}^2 + l_{22}^2 - l_{21}^2 l_{22}^2 \nabla^2) \nabla\nabla \cdot \mathbf{B}' + k D_{11}^2 D_{12}^2 \nabla \varphi \tag{60}$$

and note that, by (55) and (57),

$$D_{21}^2 D_{22}^2 \mathbf{B}'' = 0, \qquad \nabla \cdot \mathbf{B}'' = 0. \tag{61}$$

Then substitute (60) and (58) into (51) to get

$$\mathbf{u} = \mathbf{B}' + \mathbf{B}'' - (l_{21}^2 + l_{22}^2 - l_{21}^2 l_{22}^2 \nabla^2) \nabla\nabla \cdot \mathbf{B}' - \tfrac{1}{2} \nabla (D_{11}^2 D_{12}^2 - k^{-1})(\mathbf{r} \cdot D_{21}^2 D_{22}^2 \mathbf{B}' + B_0). \tag{62}$$

Finally, define

$$\mathbf{B} = \mathbf{B}' + \mathbf{B}''. \tag{63}$$

In view of (61), we may write (62) and (56) in the form of (48) and (50); and (59) is already in the form of (49). Thus, (48) is a complete solution of the equation of equilibrium, (24) or (25), if the stress functions B_0 and $\mathbf{B}$ satisfy (49) and (50).

9. CONCENTRATED FORCE

In an infinite region V, let the body force be zero outside a finite region V_0 which contains the origin and a non-vanishing field of undirectional forces $\mathbf{f}$ per unit volume. A concentrated force is defined by

$$\mathbf{F} = \lim_{V_0 \to 0} \int_V \mathbf{f}_Q \, dV_Q. \tag{64}$$

To find B_0 and $\mathbf{B}$ for the concentrated force, we have to solve equations of the type (49) and (50), viz.:

$$D_{i1}^2 D_{i2}^2 \nabla^2 \psi = \rho, \qquad i = 1, 2 \text{ (not summed)}. \tag{65}$$

This may be done by constructing the pertinent Green's formula from Green's identity

$$\int_S \mathbf{n} \cdot (\varphi_i \nabla \psi - \psi \nabla \varphi_i)_Q \, dS_Q = \int_V (\varphi_i \nabla^2 \psi - \psi \nabla^2 \varphi_i)_Q \, dV_Q \tag{66}$$

in the usual way, as follows. In (66) replace φ_i and ψ, successively, by

$$(1 - l_{i1}^2 \nabla^2)(1 - l_{i2}^2 \nabla^2)(\varphi_i, \psi),$$

$$l_{i1} l_{i2} [1 - (l_{i1}^2 + l_{i2}^2) \nabla^2] \nabla^2 (\varphi_i, \psi),$$

$$[(l_{i1}^2 + l_{i2}^2)^2 - 2 l_{i1}^2 l_{i2}^2] \tfrac{1}{2} \nabla^2 (\varphi_i, \psi),$$

$$[l_{i1}^4 l_{i2}^4 - l_{i1}^2 l_{i2}^2 (l_{i1}^2 + l_{i2}^2)^2] \tfrac{1}{4} \nabla^4 (\varphi_i, \psi)$$

and subtract the sum of the last three from the first of the resulting four equations. Now choose

$$\varphi_i(r_1) = \frac{1}{r_1} \left(1 - \frac{l_{i1}^2 e^{-r_1/l_{i1}}}{l_{i1}^2 - l_{i2}^2} + \frac{l_{i2}^2 e^{-r_1/l_{i2}}}{l_{i1}^2 - l_{i2}^2} \right), \qquad i = 1, 2$$

and note that

$$D_{i1}^2 D_{i2}^2 \nabla^2 \varphi_i = 0$$

except at P. Exclude the point P by surrounding it with a spherical surface of radius ε and center P. Upon passing to the limit as $\varepsilon \to 0$, we find the Green's formula

$$4\pi\psi = \int_S \mathbf{n} \cdot \sum_{j=1}^{j=4} A_j [\mathscr{L}_j \varphi_i \nabla \mathscr{L}_j \psi - \mathscr{L}_j \psi \nabla \mathscr{L}_j \varphi_i]_Q \, dS_Q - \int_V \rho_Q \varphi_i \, dV_Q, \tag{67}$$

where

$$\mathscr{L}_1 = D_{i1}^2 D_{i2}^2, \quad \mathscr{L}_2 = [1-(l_{i1}^2+l_{i2}^2)\nabla^2]\nabla^2, \quad \mathscr{L}_3 = \nabla^2, \quad \mathscr{L}_4 = \nabla^4,$$

$$A_1 = 1, \quad A_2 = -l_{i1}^2 l_{i2}^2, \quad A_3 = 2l_{i1}^2 l_{i2}^2 - (l_{i1}^2+l_{i2}^2)^2, \quad A_4 = l_{i1}^2 l_{i2}^2[(l_{i1}^2+l_{i2}^2)^2 - l_{i1}^2 l_{i2}^2].$$

If B_0 and $\mathbf{B}$ vanish at infinity at least as r^{-1}, the solution for the concentrated force is, from (49), (50), (64) and (67),

$$4\pi\mu B_0 = -\lim_{V_0 \to 0} \int_V [\mathbf{r}' \cdot D_{11}^2 D_{12}^2 \mathbf{f} - 4(l_{11}^2 + l_{12}^2 - \tfrac{3}{2} l_{11}^2 l_{12}^2 \nabla^2) \nabla \cdot \mathbf{f}]_Q \varphi_1(r_1) \, dV_Q, \tag{68}$$

$$4\pi\mu \mathbf{B} = \lim_{V_0 \to 0} \int_V \mathbf{f}_Q \varphi_2(r_1) \, dV_Q, \tag{69}$$

where $r' = (\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}$. Now,

$$\lim_{V_0 \to 0} r_1 = r, \qquad \lim_{V_0 \to 0} r' = 0.$$

Hence, the solution for the concentrated force is

$$2\pi\mu B_0 = \mathbf{F} \cdot \nabla(l_{11}^2 + l_{12}^2 - l_{11}^2 l_{12}^2 \nabla^2)\varphi_1(r),$$

$$4\pi\mu \mathbf{B} = \mathbf{F}\varphi_2(r).$$

The limit for $\mathbf{B}$ follows directly from (69) while that for B_0 is obtained from (68) after successive applications of the chain rule and the divergence theorem to separate $\mathbf{f}$ as a factor, of the integrand of the volume integral, before passing to the limit. Finally, it is apparent that B_0 and $\mathbf{B}$ have the required behavior at infinity.

10. ELASTIC LIQUID

The linear theory of deformation of an elastic liquid can be treated as a special case of a solid in which the potential energy-density is a function of only the infinitesimal dilatation and its gradients instead of the full infinitesimal strain and its gradients. Thus,

$$W_\Delta = W_\Delta(\Delta, \nabla\Delta, \nabla\nabla\Delta), \tag{70}$$

where $\Delta = \nabla \cdot \mathbf{u}$ is the infinitesimal dilatation. For the centrosymmetric, isotropic liquid, the form of W_Δ is obtained from (19) by discarding all but the terms indicated in (70), leaving

$$W_\Delta = \tfrac{1}{2}\lambda\Delta^2 + a_1 \nabla\Delta \cdot \nabla\Delta + b_1(\nabla^2\Delta)^2 + b_2 \nabla\nabla\Delta : \nabla\nabla\Delta + c_1 \Delta\nabla^2\Delta + b_0 \nabla^2\Delta. \tag{71}$$

This contains the term $b_2 \nabla\nabla\Delta : \nabla\nabla\Delta$ which Hart omitted in a previous study [3].

As noted by Hart, either the dilatation or the displacement may be subject to variation in varying the potential energy:

$$\delta \int_V W_\Delta \, dV = -\int_V (p\delta\Delta + \mathbf{p}\cdot\nabla\delta\Delta + \boldsymbol{\pi} : \nabla\nabla\delta\Delta)\, dV, \tag{72}$$

or

$$\delta \int_V W_\Delta \, dV = -\int_V (p\nabla\cdot\delta\mathbf{u} + \mathbf{p}\cdot\nabla\nabla\cdot\delta\mathbf{u} + \boldsymbol{\pi} : \nabla\nabla\nabla\cdot\delta\mathbf{u})\, dV, \tag{73}$$

where, in either case,

$$p = -\frac{\partial W_\Delta}{\partial\Delta} = -\lambda\Delta - c_1\nabla^2\Delta, \tag{74a}$$

$$\mathbf{p} = -\frac{\partial W_\Delta}{\partial\nabla\Delta} = -2a_1\nabla\Delta, \tag{74b}$$

$$\boldsymbol{\pi} = -\frac{\partial W_\Delta}{\partial\nabla\nabla\Delta} = -(b_0 + 2b_1\nabla^2\Delta + c_1\Delta)\mathbf{I} - 2b_2\nabla\nabla\Delta. \tag{74c}$$

We consider, first, $\delta\Delta$ as the independent variation. Then, by applying the chain rule and the divergence theorem, we transform the right-hand side of (72) to

$$-\int_V (p - \nabla\cdot\mathbf{p} + \nabla\nabla : \boldsymbol{\pi})\delta\Delta \, dV - \int_S \mathbf{n}\cdot(\mathbf{p} - \nabla\cdot\boldsymbol{\pi})\delta\Delta \, dS - \int_S \mathbf{n}\cdot\boldsymbol{\pi}\cdot\nabla\delta\Delta \, dS.$$

Hart did not observe that $\nabla\delta\Delta$ is not independent of $\delta\Delta$ on S; but, by the same procedure as in Section 2, we find

$$\int_S \mathbf{n}\cdot\boldsymbol{\pi}\cdot\nabla\delta\Delta \, dS = \int_S [\mathbf{L}\cdot(\mathbf{n}\cdot\boldsymbol{\pi})]\delta\Delta \, dS + \int_S \mathbf{nn} : \boldsymbol{\pi} D\delta\Delta \, dS,$$

thereby converting to independent variations $\delta\Delta$ and $D\delta\Delta$. Hence

$$\delta \int_V W_\Delta \, dV = -\int_V (p - \nabla\cdot\mathbf{p} + \nabla\nabla : \boldsymbol{\pi})\delta\Delta \, dV - \int_S \mathbf{n}\cdot(\mathbf{p} - \nabla\cdot\boldsymbol{\pi})\delta\Delta \, dS - \int_S [\mathbf{L}\cdot(\mathbf{n}\cdot\boldsymbol{\pi})]\delta\Delta \, dS - \int_S \mathbf{nn} : \boldsymbol{\pi} D\delta\Delta \, dS. \tag{75}$$

The principle of stationary potential energy now assumed is

$$\delta \int_V W_\Delta \, dV = \int_V \varphi\delta\Delta \, dV + \int_S (\overset{1}{t}_\Delta \delta\Delta + \overset{2}{t}_\Delta D\delta\Delta)\, dS, \tag{76}$$

where φ is the potential of a body force per unit volume and $\overset{1}{t}_\Delta$, $\overset{2}{t}_\Delta$ are generalized surface-tractions. Equating the right-hand sides of (75) and (76), we find the stress-equation of equilibrium

$$p - \nabla\cdot\mathbf{p} + \nabla\nabla : \boldsymbol{\pi} + \varphi = 0 \tag{77}$$

and the traction boundary conditions

$$\overset{1}{t}_\Delta = -\mathbf{n}\cdot(\mathbf{p} - \nabla\cdot\boldsymbol{\pi}) - \mathbf{L}\cdot(\mathbf{n}\cdot\boldsymbol{\pi}), \qquad \overset{2}{t}_\Delta = -\mathbf{nn} : \boldsymbol{\pi}. \tag{78}$$

The equation of equilibrium on Δ is obtained by substituting (74) into (77), with the result:

$$\lambda\Delta - 2(a_1 - c_1)\nabla^2\Delta + 2(b_1 + b_2)\nabla^4\Delta = \varphi, \tag{79}$$

or

$$\lambda(1-l_1^2\nabla^2)(1-l_2^2\nabla^2)\Delta = \varphi, \tag{80}$$

where

$$\lambda l_i^2 = a_1-c_1 \pm [(a_1-c_1)^2-2\lambda(b_1+b_2)]^{\frac{1}{2}}, \qquad i = 1,2. \tag{81}$$

We now turn to (73), in which $\delta\mathbf{u}$ is the variation, and find, by application of the chain rule and the divergence theorem,

$$\delta\int_V W_\Delta \,dV = \int_V [\nabla(p-\nabla\cdot\mathbf{p}+\nabla\nabla:\boldsymbol{\pi})]\cdot\delta\mathbf{u}\,dV - \int_S (p-\nabla\cdot\mathbf{p}+\nabla\nabla:\boldsymbol{\pi})\mathbf{n}\cdot\delta\mathbf{u}\,dS$$
$$-\int_S [\mathbf{n}\cdot(\mathbf{p}-\nabla\cdot\boldsymbol{\pi})]\nabla\cdot\delta\mathbf{u}\,dS - \int_S \mathbf{n}\cdot\boldsymbol{\pi}\cdot\nabla\nabla\cdot\delta\mathbf{u}\,dS. \tag{82}$$

Here, again, Hart did not notice the lack of independence of $\delta\mathbf{u}$, $\nabla\cdot\delta\mathbf{u}$ and $\nabla\nabla\cdot\delta\mathbf{u}$ on S; but, by the same procedure as in Section 2, we find

$$\delta\int_V W_\Delta \,dV = \int_V [\nabla(p-\nabla\cdot\mathbf{p}+\nabla\nabla:\boldsymbol{\pi})]\cdot\delta\mathbf{u}\,dV - \int_S (p-\nabla\cdot\mathbf{p}+\nabla\nabla:\boldsymbol{\pi})\mathbf{n}\cdot\delta\mathbf{u}\,dS$$
$$-\int_S \{\mathbf{L}[\mathbf{n}\cdot(\mathbf{p}-\nabla\cdot\boldsymbol{\pi})+\mathbf{L}\cdot(\mathbf{n}\cdot\boldsymbol{\pi})]-\mathbf{L}\cdot[(\overset{s}{\nabla}\mathbf{n})\mathbf{nn}:\boldsymbol{\pi}]\}\cdot\delta\mathbf{u}\,dS$$
$$-\int_S \{[\mathbf{n}\cdot(\mathbf{p}-\nabla\cdot\boldsymbol{\pi})+\mathbf{L}\cdot(\mathbf{n}\cdot\boldsymbol{\pi})]\mathbf{n}+\mathbf{L}(\mathbf{nn}:\boldsymbol{\pi})\}\cdot D\delta\mathbf{u}\,dS$$
$$-\int_S (\mathbf{nn}:\boldsymbol{\pi})\mathbf{n}\cdot D^2\delta\mathbf{u}\,dS. \tag{83}$$

In this case, the principle of stationary potential energy is assumed to be:

$$\delta\int_V W_\Delta \,dV = \int_V \mathbf{f}\cdot\delta\mathbf{u}\,dV + \int_S (\overset{1}{\mathbf{t}}\cdot\delta\mathbf{u}+\overset{2}{\mathbf{t}}\cdot D\delta\mathbf{u}+\overset{3}{\mathbf{t}}\cdot D^2\delta\mathbf{u})\,dS, \tag{84}$$

so that the stress-equation of equilibrium is

$$\nabla(p-\nabla\cdot\mathbf{p}+\nabla\nabla:\boldsymbol{\pi})-\mathbf{f} = 0 \tag{85}$$

and the traction boundary conditions are

$$\overset{1}{\mathbf{t}} = -\mathbf{n}(p-\nabla\cdot\mathbf{p}+\nabla\nabla:\boldsymbol{\pi})-\mathbf{L}[\mathbf{n}\cdot(\mathbf{p}-\nabla\cdot\boldsymbol{\pi})+\mathbf{L}\cdot(\mathbf{n}\cdot\boldsymbol{\pi})]+\mathbf{L}\cdot[(\overset{s}{\nabla}\mathbf{n})\mathbf{nn}:\boldsymbol{\pi}], \tag{86a}$$

$$\overset{2}{\mathbf{t}} = -\mathbf{nn}\cdot(\mathbf{p}-\nabla\cdot\boldsymbol{\pi})-\mathbf{nL}\cdot(\mathbf{n}\cdot\boldsymbol{\pi})-\mathbf{L}(\mathbf{nn}:\boldsymbol{\pi}), \tag{86b}$$

$$\overset{3}{\mathbf{t}} = -\mathbf{nnn}:\boldsymbol{\pi}. \tag{86c}$$

The displacement-equation of equilibrium, obtained by substituting (74) in (85) and employing (81), is

$$\lambda(1-l_1^2\nabla^2)(1-l_2^2\nabla^2)\nabla\nabla\cdot\mathbf{u}+\mathbf{f} = 0. \tag{87}$$

The formula for the surface-tension is obtained as in Section 4:

$$T = \tfrac{1}{2}b_0\mathbf{n}\cdot\nabla\Delta|_S \tag{88}$$

and is the same for variation of Δ as for variation of $\mathbf{u}$.

11. DILATATION AND SURFACE-TENSION AT A PLANE SURFACE OF A LIQUID

The problem of the liquid half-space $x \geqslant 0$, with traction-free boundary and no body force, is formulated, first, in terms of the equations derived from variation of Δ. The solution of (80), vanishing as $x \to \infty$, with Δ a function of x only and $\varphi = 0$, is

$$\Delta = A_1 e^{-x/l_1} + A_2 e^{-x/l_2}; \tag{89}$$

and the two boundary conditions (78) become

$$[(2a_1 - c_1)\Delta' - 2(b_1 + b_2)\Delta''']_{x=0} = 0, \qquad [c_1\Delta + 2(b_1 + b_2)\Delta'']_{x=0} = -b_0, \tag{90}$$

where primes signify differentiations with respect to x. Substituting (89) in (90), we have a pair of equations for A_1 and A_2—the solution of which, after the use of (81), is

$$A_1 = -\frac{b_0 l_1 (l_2^2 + l_0^2)}{\lambda[l_1(l_2^2 + l_0^2)^2 - l_2(l_1^2 + l_0^2)^2]},$$

$$A_2 = \frac{b_0 l_2 (l_1^2 + l_0^2)}{\lambda[l_1(l_2^2 + l_0^2)^2 - l_2(l_1^2 + l_0^2)^2]},$$

where $l_0^2 = c_1/\lambda$. Then (88) and (89) yield the formula for the surface-tension at a plane, liquid surface:

$$T_0 = \frac{b_0^2 (l_1^2 - l_2^2)}{2\lambda[l_1(l_2^2 + l_0^2)^2 - l_2(l_1^2 + l_0^2)^2]}. \tag{91}$$

In the alternate case of the formulation derived from varying $\mathbf{u}$, the appropriate solution of (87) may be written in the form

$$u_x = A_0 - l_1 A_1' e^{-x/l_1} - l_2 A_2' e^{-x/l_2}. \tag{92}$$

There are now the three boundary conditions (86); but they lead to

$$A_0 = 0, \qquad A_1' = A_1, \qquad A_2' = A_2$$

so that the two solutions are the same.

12. DILATATION AND SURFACE-TENSION AT A SPHERICAL SURFACE OF A LIQUID

We consider a liquid sphere under no body force and with a traction-free surface of radius r_0. The appropriate solution of the equation of equilibrium (80), regular at the origin $r = 0$, can be written in the form

$$\Delta = A_1 l_1 r^{-1} \sinh(r/l_1) + A_2 l_2 r^{-1} \sinh(r/l_2). \tag{93}$$

Then the boundary conditions (78) yield

$$\sum_{i=1}^{i=2} A_i r_i [2(b_1 + b_2) - (2a_1 - c_1) l_i^2](r_i \cosh r_i - \sinh r_i) = 0, \tag{94a}$$

$$\sum_{i=1}^{i=2} A_i \{r_i [2b_2(1 + 2r_i^{-1}) + 2b_1 + c_1 l_i^2] \sinh r_i - 4b_2 \cosh r_i\} = -b_0 r_0^2, \tag{94b}$$

where $r_i = r_0/l_i$. Again, it may be verified that the same result is obtained from the equations derived by varying the displacement.

To find a simple result for the first order effect of the curvature of the surface, assume that the radius of curvature is large in comparison with the l_i (i.e. $r_i \gg 1$). Then the solution of (94) is

$$A_1 \approx -\frac{2b_0\lambda^{-1}r_0(l_2^2+l_0^2)e^{-r_1}}{l_1(l_2^2+l_0^2)^2-l_2(l_1^2+l_0^2)^2+4b_2\lambda^{-1}r_0^{-1}(l_1^2-l_2^2)},$$

$$A_2 \approx \frac{2b_0\lambda^{-1}r_0(l_1^2+l_0^2)e^{-r_2}}{l_1(l_2^2+l_0^2)^2-l_2(l_1^2+l_0^2)^2+4b_2\lambda^{-1}r_0^{-1}(l_1^2-l_2^2)};$$

and the surface tension is, to the first order in l_i/r_0,

$$T \approx \frac{T_0}{1+(8b_2T_0/b_0^2r_0)}, \tag{95}$$

where T_0 is the surface-tension at a plane surface as given by (91). The condition of positive definiteness of the quadratic part of the potential energy-density (71) requires $b_2 > 0$. Hence, as the radius of the surface diminishes, the surface-tension becomes algebraically smaller.

A result of the classical theory of surface-tension is that the internal pressure in the sphere is $2T/r_0$. In the present type of theory, a variety of quantities could be assigned the role of internal pressure. We could choose p or $\overset{1}{t}_\theta$; but $\overset{1}{t}_\theta$ could take a variety of forms for various choices of independent variables and independent variations. Internal pressure, along with stress, appears to have lost its former significance owing to the consideration of long range forces. However, dilatation is defined uniquely and can serve as a basis of comparison of the two theories. In the classical case, the dilatation in the sphere would be uniform and proportional to T/r_0. In the present theory, the dilatation is non-uniform; but we may calculate the *average* dilatation:

$$\Delta_{\text{ave}} = \frac{3}{4\pi r_0^3}\int_V \Delta\, dV,$$

$$= \frac{3}{r_0^2}\sum_{i=1}^{i=2} A_i l_i^2(\cosh r_i - r_i^{-1}\sinh r_i).$$

Assuming, again, that $r_i \gg 1$, we find, to the first order in l_i/r_0,

$$\Delta_{\text{ave}} \approx -\frac{6c_1}{b_0\lambda}\frac{T}{r_0},$$

where T is given by (95). Thus, as in the classical theory, the average dilatation in the sphere is directly proportional to the surface-tension and inversely proportional to the radius.

Now consider a spherical *cavity* of radius r_0. For this we take the solution of (80), with $\varphi = 0$, in the form

$$\Delta = A_1 l_1 r^{-1}e^{-r/l_1} + A_2 l_2 r^{-1}e^{-r/l_2}. \tag{96}$$

The conditions (78), for a free boundary, require

$$\sum_{i=1}^{i=2} A_i[2(b_1+b_2)-(2a_1-c_1)l_i^2](1+r_i)r_i e^{-r_i} = 0, \tag{97a}$$

$$\sum_{i=1}^{i=2} A_i r_i[2b_2(1+2r_i^{-1}+2r_i^{-2})+2b_1+c_1 l_i^2]e^{-r_i} = -b_0 r_0^2. \tag{97b}$$

Again we consider the case $r_i \gg 1$ and find, to the first order in l_i/r_0,

$$T = \frac{T_0}{1-(8b_2T_0/b_0^2r_0)}. \tag{98}$$

Thus, for both convex and concave spherical surfaces, the surface-tension, to the first order in l_i/r_0, is

$$T = \frac{T_0}{1+8b_0^{-2}b_2\kappa T_0} \tag{99}$$

where κ is the curvature: positive for the sphere and negative for the cavity.

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APPENDIX

Boundary conditions at edges and corners

In Sections 2 and 10, it was assumed that the surface S is smooth. Additional boundary conditions are required if S has edges and corners.

Suppose the surface has an edge, C, formed by the intersection of two segments, S_1 and S_2, of S. Then, for each segment, the surface divergence theorem [8] is

$$\int_{S_i} \overset{s}{\nabla} \cdot \mathbf{f}\, dS = \int_{S_i} (\overset{s}{\nabla} \cdot \mathbf{n})\mathbf{n} \cdot \mathbf{f}\, dS + \oint_C \mathbf{m}_i \cdot \mathbf{f}_i\, ds, \qquad i = 1, 2, \qquad \text{(not summed)}$$

where s is measured along C in the direction of its unit tangent $\mathbf{s}_i$; and $\mathbf{m}_i$ $(= \mathbf{s}_i \times \mathbf{n}_i)$ is the unit outward normal to C tangent to S_i. The contributions of the line integrals have to be added to the previous results. Thus, to (8) and (10), we have to add

$$\oint_C [\mathbf{m} \cdot \boldsymbol{\Phi} \cdot \mathbf{v}]\, ds, \tag{8A}$$

and

$$\oint_C [\mathbf{m} \cdot \boldsymbol{\Psi} : \boldsymbol{\Phi}]\, ds, \tag{10A}$$

where the boldfaced brackets [ ] denote the value of the enclosed quantity on C when approached over S_1 minus its value when approached over S_2; and ds is positive in the direction of $\mathbf{s}_1$.

Considering, first, the case of the solid, we must add to (11), (12) and (13):

$$\oint_C [\mathbf{mn} : (\overset{2}{\tau} - \nabla \cdot \overset{3}{\tau})] \cdot \delta\mathbf{u}\, ds, \tag{11A}$$

$$\oint_C [\mathbf{mn} : \overset{3}{\tau} : \nabla\delta\mathbf{u}]\, ds, \tag{12A*}$$

$$\oint_C [\mathbf{m} \cdot [\mathbf{L} \cdot (\mathbf{n} \cdot \overset{3}{\tau})]] \cdot \delta\mathbf{u}\, ds, \tag{13A}$$

respectively, while to (14) we have to add

$$-\oint_C [\mathbf{m} \cdot (\overset{s}{\nabla}\mathbf{n}) \cdot (\mathbf{nn} : \overset{3}{\tau})] \cdot \delta\mathbf{u}\, ds \tag{14A1}$$

and

$$\oint_C [\mathbf{mnn} : \overset{3}{\tau} \cdot D\delta\mathbf{u}]\, ds. \tag{14A*}$$

The integrands in (12A*) and (14A*) require resolution into terms with coefficients that are independent variations. For (14A*) we write

$$\oint_C (\mathbf{nnm} : \overset{3}{\tau})_1 D_{n_1}\delta\mathbf{u}\, ds - \oint_C (\mathbf{nnm} : \overset{3}{\tau})_2 D_{n_2}\delta\mathbf{u}\, ds, \tag{14A2}$$

where $D_{n_i}\delta\mathbf{u}$ and $(\mathbf{nnm} : \overset{3}{\tau})_i$ are the limits at C of $\mathbf{n} \cdot \nabla\delta\mathbf{u}$ and $\mathbf{nnm} : \overset{3}{\tau}$ approached over S_i. As for (12A*), we first write

$$\nabla\delta\mathbf{u} = \mathbf{nn} \cdot \nabla\delta\mathbf{u} + \mathbf{mm} \cdot \nabla\delta\mathbf{u} + \mathbf{ss} \cdot \nabla\delta\mathbf{u} = \mathbf{n}D_n\delta\mathbf{u} + \mathbf{m}D_m\delta\mathbf{u} + \mathbf{s}D_s\delta\mathbf{u}.$$

Then (12A*) is the sum of

$$\oint_C (\mathbf{nmn} : \overset{3}{\boldsymbol{\tau}})_1 \cdot D_{n_1}\delta\mathbf{u}\, ds - \oint_C (\mathbf{nmn} : \overset{3}{\boldsymbol{\tau}})_2 \cdot D_{n_2}\delta\mathbf{u}\, ds$$

$$+ \oint_C (\mathbf{nmm} : \overset{3}{\boldsymbol{\tau}})_1 \cdot D_{m_1}\delta\mathbf{u}\, ds - \oint_C (\mathbf{nmm} : \overset{3}{\boldsymbol{\tau}})_2 \cdot D_{m_2}\delta\mathbf{u}\, ds \tag{12A1}$$

and

$$\oint_C [\mathbf{nm} : \overset{3}{\boldsymbol{\tau}} : \mathbf{s}D_s\delta\mathbf{u}]\, ds = \oint_C [D_s(\mathbf{nms} : \overset{3}{\boldsymbol{\tau}} \cdot \delta\mathbf{u})]\, ds - \oint_C [D_s(\mathbf{nms} : \overset{3}{\boldsymbol{\tau}})] \cdot \delta\mathbf{u}\, ds$$

$$= [|\mathbf{nms} : \overset{3}{\boldsymbol{\tau}}|^{+}_{-}] \cdot \delta\mathbf{u} - \oint_C [D_s(\mathbf{nms} : \overset{3}{\boldsymbol{\tau}})] \cdot \delta\mathbf{u}\, ds, \tag{12A2}$$

where the quantity $[|\mathbf{nms} : \overset{3}{\boldsymbol{\tau}}|^{+}_{-}]$ is the difference between discontinuities in $(\mathbf{nms} : \overset{3}{\boldsymbol{\tau}})_1$ and $(\mathbf{nms} : \overset{3}{\boldsymbol{\tau}})_2$ along C or at a corner of C.

Altogether, the additional terms are (11A), (12A1), (12A2), (13A), (14A1) and (14A2), so that we have to add to (15) and (16):

$$\oint_C [\mathbf{mn} : (\overset{2}{\boldsymbol{\tau}} - \nabla \cdot \overset{3}{\boldsymbol{\tau}}) + \mathbf{m} \cdot [\mathbf{L} \cdot (\mathbf{n} \cdot \overset{3}{\boldsymbol{\tau}})] - D_s(\mathbf{nms} : \overset{3}{\boldsymbol{\tau}}) - \mathbf{m} \cdot (\overset{4}{\nabla}\mathbf{n}) \cdot (\mathbf{nn} : \overset{3}{\boldsymbol{\tau}})] \cdot \delta\mathbf{u}\, ds$$

$$+ \oint_C 2(\mathbf{nmn} : \overset{3}{\boldsymbol{\tau}})_1 \cdot D_{n_1}\delta\mathbf{u}\, ds - \oint_C 2(\mathbf{nmn} : \overset{3}{\boldsymbol{\tau}})_2 \cdot D_{n_2}\delta\mathbf{u}\, ds$$

$$+ \oint_C (\mathbf{mnm} : \overset{3}{\boldsymbol{\tau}})_1 \cdot D_{m_1}\delta\mathbf{u}\, ds - \oint_C (\mathbf{mnm} : \overset{3}{\boldsymbol{\tau}})_2 \cdot D_{m_2}\delta\mathbf{u}\, ds$$

$$+ [|\mathbf{mns} : \overset{3}{\boldsymbol{\tau}}|^{+}_{-}] \cdot \delta\mathbf{u} \tag{15A}$$

and

$$\oint_C \mathbf{F} \cdot \delta\mathbf{u}\, ds + \oint_C (\mathbf{N}_1 \cdot D_{n_1}\delta\mathbf{u} + \mathbf{N}_2 \cdot D_{n_2}\delta\mathbf{u} + \mathbf{T}_1 \cdot D_{m_1}\delta\mathbf{u} + \mathbf{T}_2 \cdot D_{m_2}\delta\mathbf{u})\, ds + \mathbf{G} \cdot \delta\mathbf{u}, \tag{16A}$$

respectively. Hence, equating (15A) and (16A), we have the edge and corner conditions

$$\mathbf{F} = [\mathbf{mn} : (\overset{2}{\boldsymbol{\tau}} - \nabla \cdot \overset{3}{\boldsymbol{\tau}}) + \mathbf{m} \cdot [\mathbf{L} \cdot (\mathbf{n} \cdot \overset{3}{\boldsymbol{\tau}})] - D_s(\mathbf{nms} : \overset{3}{\boldsymbol{\tau}}) - \mathbf{m} \cdot (\overset{4}{\nabla}\mathbf{n}) \cdot (\mathbf{nn} : \overset{3}{\boldsymbol{\tau}})],$$

$$\mathbf{N}_1 = 2(\mathbf{nmn} : \overset{3}{\boldsymbol{\tau}})_1,$$

$$\mathbf{N}_2 = -2(\mathbf{nmn} : \overset{3}{\boldsymbol{\tau}})_2,$$

$$\mathbf{T}_1 = (\mathbf{nmm} : \overset{3}{\boldsymbol{\tau}})_1,$$

$$\mathbf{T}_2 = -(\mathbf{nmm} : \overset{3}{\boldsymbol{\tau}})_2,$$

$$\mathbf{G} = [|\mathbf{mns} : \overset{3}{\boldsymbol{\tau}}|^{+}_{-}].$$

As in the case of the boundary conditions on a smooth surface, alternative edge and corner conditions involving displacements are apparent from the form of (16A).

Turning, now, to the boundary conditions for the liquid, we find that, when it is the dilatation that is varied, we have only to add to (75) and (76):

$$-\oint_C [\mathbf{mn} : \boldsymbol{\pi}]\delta\Delta \, ds \tag{75A}$$

and

$$\oint_C \overset{0}{t}_{\Delta}\delta\Delta \, ds, \tag{76A}$$

respectively; so that the only boundary condition to be added to (78) is

$$\overset{0}{t}_{\Delta} = -[\mathbf{mn} : \boldsymbol{\pi}]. \tag{78A}$$

On the other hand, when it is the displacement that is varied, the additional terms are similar to those for the solid. To (83) and (84) we have to add

$$-\oint_C [\mathbf{n}\cdot(\mathbf{p}-\nabla\cdot\boldsymbol{\pi})\mathbf{m}+\mathbf{mL}\cdot(\mathbf{n}\cdot\boldsymbol{\pi})-D_s(\mathbf{mn} : \boldsymbol{\pi}\mathbf{s})-\mathbf{nn} : \boldsymbol{\pi}\mathbf{m}\cdot(\overset{s}{\nabla}\mathbf{n})]\cdot\delta\mathbf{u} \, ds$$

$$-\oint_C (\mathbf{nn} : \boldsymbol{\pi}\mathbf{m}+\mathbf{mn} : \boldsymbol{\pi}\mathbf{n})_1 \cdot D_{n_1}\delta\mathbf{u} \, ds+\oint_C (\mathbf{nn} : \boldsymbol{\pi}\mathbf{m}+\mathbf{mn} : \boldsymbol{\pi}\mathbf{n})_2 \cdot D_{n_2}\delta\mathbf{u} \, ds$$

$$-\oint_C (\mathbf{mn} : \boldsymbol{\pi}\mathbf{m})_1 \cdot D_{m_1}\delta\mathbf{u} \, ds+\oint_C (\mathbf{mn} : \boldsymbol{\pi}\mathbf{m})_2 \cdot D_{m_2}\delta\mathbf{u} \, ds-[|\mathbf{mn} : \boldsymbol{\pi}\mathbf{s}|^{+}_{-}]\cdot\delta\mathbf{u} \tag{83A}$$

and

$$\oint_C \mathbf{F}\cdot\delta\mathbf{u} \, ds+\oint_C (\mathbf{N}_1\cdot D_{n_1}\delta\mathbf{u}+\mathbf{N}_2\cdot D_{n_2}\delta\mathbf{u}+\mathbf{T}_1\cdot D_{m_1}\delta\mathbf{u}+\mathbf{T}_2\cdot D_{m_2}\delta\mathbf{u}) \, ds+\mathbf{G}\cdot\delta\mathbf{u}, \tag{84A}$$

respectively. Then, upon equating (83A) and (84A) we find the edge and corner conditions for the liquid:

$$\begin{aligned}
\mathbf{F} &= -[\mathbf{n}\cdot(\mathbf{p}-\nabla\cdot\boldsymbol{\pi})\mathbf{m}+\mathbf{mL}\cdot(\mathbf{n}\cdot\boldsymbol{\pi})-D_s(\mathbf{mn} : \boldsymbol{\pi}\mathbf{s})-\mathbf{nn} : \boldsymbol{\pi}\mathbf{m}\cdot(\overset{s}{\nabla}\mathbf{n})],\\
\mathbf{N}_1 &= -(\mathbf{nn} : \boldsymbol{\pi}\mathbf{m}+\mathbf{mn} : \boldsymbol{\pi}\mathbf{n})_1,\\
\mathbf{N}_2 &= (\mathbf{nn} : \boldsymbol{\pi}\mathbf{m}+\mathbf{mn} : \boldsymbol{\pi}\mathbf{n})_2,\\
\mathbf{T}_1 &= -(\mathbf{mn} : \boldsymbol{\pi}\mathbf{m})_1,\\
\mathbf{T}_2 &= (\mathbf{mn} : \boldsymbol{\pi}\mathbf{m})_2,\\
\mathbf{G} &= -[|\mathbf{mn} : \boldsymbol{\pi}\mathbf{s}|^{+}_{-}].
\end{aligned}$$

Again, alternative edge and corner conditions involving displacements are apparent from the form of (84A).

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Zusammenfassung—In diesem Aufsatz wird die lineare Deformationstheorie eines elastischen Festkörpers formuliert, bei welchem die potentielle Energiedichte eine Funktion der Beanspruchung und deren Gradienten erster und zweiter Ordnung ist. In diese Theorie sind die Kohäsionskraft und Oberflächenbeanspruchung als wesentliche Merkmale eingebaut. Eine Lösung für die Beanspruchung und Oberflächenspannung pro Flächeneinheit, wie sie bei der in einer Ebene liegenden Trennung eines Festkörpers auftreten, wird angegeben, und ein Vergleich mit dem analogen Gittermodell angestellt. Ausserdem wird die allgemeine Lösung der Gleichgewichtsverschiebegleichung angegeben, und zwar drückt man diese in Spannungsfunktionen und mit Hilfe der besonderen Lösung für die konzentrierte Kraft aus. Der Spezialfall einer Flüssigkeit wird erwogen, und die Lösungen für die Oberflächenspannung bei ebenen und runden Oberflächen werden angegeben.

Абстракт—В настоящей статье формулируется линейная теория деформации упругого твердого тела, в котором потенциальная энергия-плотность является функцией деформации и ее первым и вторым градиентом. В этой теории сила сцепления и поверхностное натяжение являются внутренними силами. Дано решение для деформации и поверхностного натяжения, или поверхностной энергии на единицу площади, полученной в разделении твердого тела по плоскости; сделано сравнение с аналогичной решетчатой моделью. Также предлагается общее решение уравнения смещения при равновесии в условиях функции напряжения и особое решение для сосредоточенной силы.

Рассматривается специальный случай жидкости и дается решение для поверхностного натяжения в плоской и сферической поверхностях.